Matb41 Week 9 Notes

1. Extreme Value Theorem (EVT):
   - Def: Let D be a compact (Bounded and Closed) set in $\mathbb{R}^n$ and let $f: D \subset \mathbb{R}^n \to \mathbb{R}$ be cont. Then f has both a global max and global min on D.

   - Steps on how to find global ext(s) of continuous functions on a compact set:

     Let $f: \mathbb{R}^n \to \mathbb{R}$ be cont on a compact set D.
     1. Find the crit points of f on the interior of D.
     2. Find the crit points of f restricted to ∂D (The Boundary).
     3. Compute f at each of these crit points.
     4. Compare and choose the biggest/smallest.

- E.g. Find the global max and min values of $f(x,y) = x^2 + y^2 - 2x + 2y + 5$ on the set $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$

Soln:

1. Interior of D: $x^2 + y^2 < 4$

$$\frac{\partial f}{\partial x} = 2x - 2 = 0 \longrightarrow x = 1$$

$$\frac{\partial f}{\partial y} = 2x + 2 = 0 \longrightarrow y = -1$$

$(1,-1) \in$ Interior of D

$(1)^2 + (-1)^2 = 2 < 4$

$\therefore (1,-1)$ is a crit point.

2. ∂D: $x^2 + y^2 = 4$

$$\left.\begin{array}{l} \text{Let } x = 2\cos\theta \\ \text{Let } y = 2\sin\theta \end{array}\right\} 0 \leq \theta \leq 2\pi$$

$$f(2\cos\theta, 2\sin\theta) = 4\cos^2\theta + 4\sin^2\theta - 4\cos\theta + 4\sin\theta + 5$$

$$= 4\sin\theta - 4\cos\theta + 9 = g(\theta)$$

$$\frac{dg}{d\theta} = 4\cos\theta + 4\sin\theta = 0$$

$\tan\theta = -1$

$\therefore \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

f has 2 crit points on ∂D

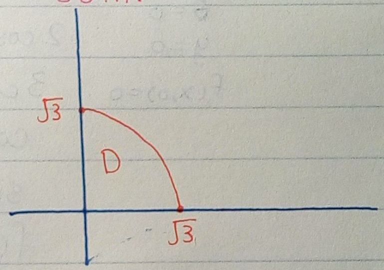
1. $(2\cos(\frac{3\pi}{4}),\ 2\sin(\frac{3\pi}{4})) = (-\sqrt{2}, \sqrt{2})$
2. $(2\cos(\frac{7\pi}{4}),\ 2\sin(\frac{7\pi}{4})) = (\sqrt{2}, -\sqrt{2})$

The 3 crit points are

1. $(1,-1)$ $f(1,-1) = 3$ Global Min
2. $(-\sqrt{2}, \sqrt{2})$ $f(-\sqrt{2},\sqrt{2}) = 9+4\sqrt{2}$ Global Max
3. $(\sqrt{2}, -\sqrt{2})$ $f(\sqrt{2},-\sqrt{2}) = 9-4\sqrt{2} \approx 3.3$

- E.g. Find the global max and min values for $f(x,y) = xy^2$ on the set $D = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2+y^2 \leq 3\}$

Soln:



1. Interior of D: $x>0,\ y>0,\ x^2+y^2<3$

$$\frac{\partial f}{\partial x} = y^2 = 0 \longrightarrow y=0$$

$$\frac{\partial f}{\partial y} = 2xy = 0 \longrightarrow y=0 \text{ or } x=0$$

$(0,0) \notin$ Interior of D

$\therefore$ There are no crit points in the interior.

2. ∂D: $x=0$, $x^2+y^2=3$

$0 \le x \le \sqrt{3}$, $0 \le y \le \sqrt{3}$

$\left.\begin{array}{l} f(0,y)=0 \\ f(x,0)=0 \end{array}\right\}$ Global Min

$\left.\begin{array}{l} \text{Let } x=\sqrt{3}\cos\theta \\ \text{Let } y=\sqrt{3}\sin\theta \end{array}\right\}$ $0 \le \theta \le \frac{\pi}{2}$

$f(\sqrt{3}\cos\theta, \sqrt{3}\sin\theta) = 3\sqrt{3}\sin^2\theta\cos\theta = g(\theta)$

$$\frac{dg}{d\theta} = 6\sqrt{3}\sin\theta\cos^2\theta - 3\sqrt{3}\sin^3\theta$$

$$= 3\sqrt{3}\sin\theta(2\cos^2\theta - \sin^2\theta) = 0$$

$\sin\theta = 0$

$\theta = 0$

$y = 0$

$f(x,0) = 0$

$2\cos^2\theta - \sin^2\theta = 0$

$2\cos^2\theta - 1 + \cos^2\theta = 0$

$3\cos^2\theta = 1$

$\cos\theta = \frac{1}{\sqrt{3}}$

$\sin\theta = \sqrt{\frac{2}{3}}$

$f(x,y) = f\left(\sqrt{3}\cdot\frac{1}{\sqrt{3}}, \sqrt{3}\cdot\sqrt{\frac{2}{3}}\right)$

$= f(1, \sqrt{2})$

$= 2$

The global min value is 0 at the boundary $x=0$, $y=0$.

The global max value is 2 at the boundary $x=1$, $y=\sqrt{2}$.

2. Lagrange Multiplier:

- Used to find the max and min values of f subject to constraints.

In our first example, $x^2 + y^2 \leq 4$ is a constraint.

- Thm: Let $f: D \subset \mathbb{R}^n \to \mathbb{R}$ and $g: D \subset \mathbb{R}^n \to \mathbb{R}$ be of class C^1, and let $x_0 \in D$ and $g(x_0) = c$. Let $S = \{x \in D \mid g(x) = c\}$. Assume that $\nabla g(x_0) \neq 0$. If x_0 is an ext of f on S, then there is a real number λ s.t. $\nabla f(x_0) = \lambda \nabla g(x_0)$.

Let $x = x(t)$ be an arbitrary curve in S that passes through $x_0 = x(t_0)$.

1. $g(x(t)) = c$

Tangent Direction $x(t)$

$$\frac{dg}{dt} = \nabla g \cdot x'(t) = \frac{dc}{dt} = 0$$

$\nabla g(x_0) \cdot x'(t_0) = 0 \Rightarrow \nabla g(x_0) \perp$ Tangent Plane of S at $x_0 = x(t_0)$

2. $f(x(t))$ has a local ext at $x_0 = x(t_0)$

$$\left.\frac{df}{dt}\right|_{t=t_0} = 0 = \nabla f(x_0) \cdot x'(t_0)$$

Since $x = x(t)$ is an arbitrary curve on S, $\nabla f(x_0) \perp$ Tangent plane of S at $x_0 = x(x_0)$.

$\therefore \nabla g(x_0) \parallel \nabla f(x_0)$

- Thm: If f, when constrained to a surface S, has a max or min at x_0, then $\nabla f(x_0) \perp S$.

- Let $a = (a_1, a_2, \dots a_n) \in R^n$ be an ext for $f: R^n \to R$ subject to the constraint $g(x_1, x_2, \dots x_n) = c$. To find the coordinates of a, we solve the system:
  1. $\nabla f(a) = \lambda \nabla g(a)$
  2. $g(a) - c = 0$

- λ is called a Lagrange Multiplier, "a" gives a constrained crit point and the process is called The Method of Lagrange Multiplier.

- Lagrange Multiplier Strategy for finding absolute extrema on regions with boundary:
  1. Construct a new function $L: R^{n+1} \to R$ by $L(x, \lambda) = f(x) - \lambda(g(x) - c)$
     L is called the Lagrange Function or the Lagrangian.

  2. Finding all the crit points of L about λ and the constrained crit points of f.

3. Evaluate all the constrained crit points of f. The largest is the max value of f and the smallest is the min value of f.

E.g. Find the constrained crit points of the func $f(x,y) = x^2+y^2-2x+2y+5$ on the set $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4\}$.

Soln:

Let $g(x,y) = x^2+y^2 = 4 = c$

$$L = x^2+y^2-2x+2y+5-\lambda(x^2+y^2-4)$$

$$\frac{\partial L}{\partial x} = 2x-2\lambda x-2 = 0 \longrightarrow 2x(1-\lambda) = 2 \quad (1)$$

$$\frac{\partial L}{\partial y} = 2y+2-2\lambda y = 0 \longrightarrow 2y(1-\lambda) = -2 \quad (2)$$

$$\frac{\partial L}{\partial \lambda} = -x^2-y^2+4 = 0 \longrightarrow x^2+y^2 = 4 \quad (3)$$

From 1 and 2, we see that $x=-y$.
Plugging $(-x)$ for y in 3, we get
$2x^2 = 4 \rightarrow x = \pm\sqrt{2}$
$y = \mp\sqrt{2}$

$\therefore$ The 2 crit points on ∂D are

1. $(\sqrt{2}, -\sqrt{2})$
2. $(-\sqrt{2}, \sqrt{2})$

E.g. Find the shortest distance from $(1,1,1)$ to $x+y-z=5$.

Soln:

$$d(x,y,z) = \frac{|Ax+By+Cz-D|}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{|1+1-1-5|}{\sqrt{1^2+1^2+1^2}}$$

$$= \frac{4}{\sqrt{3}}$$

Consider $(x,y,z) \in \mathbb{R}^3$ on the plane.

$d(x,y,z) = \sqrt{(x-1)^2+(y-1)^2+(z-1)^2}$

Let $f(x,y,z) = (x-1)^2+(y-1)^2+(z-1)^2$

Let $g(x,y,z) = x+y-z=5=c$

$$L = f(x,y,z) - \lambda g(x,y,z)$$
$$= (x-1)^2+(y-1)^2+(z-1)^2 - \lambda(x+y-z-5)$$

$$\frac{\partial L}{\partial x} = 2(x-1) - \lambda = 0 \rightarrow 2(x-1) = \lambda$$

$$\frac{\partial L}{\partial y} = 2(y-1) - \lambda = 0 \rightarrow 2(y-1) = \lambda$$

$$\frac{\partial L}{\partial z} = 2(z-1) + \lambda = 0 \longrightarrow 2(z-1) = -\lambda$$

$$\frac{\partial L}{\partial \lambda} = x + y - z = 5$$

$x = y$

$z = 2 - y$

$2y - (2-y) = 5$

$y = 7/3$

$x = 7/3$

$z = -\frac{1}{3}$

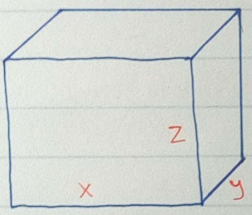
Only 1 crit point: $(7/3, 7/3, -\frac{1}{3})$

$$d\left(\frac{7}{3}, \frac{7}{3}, -\frac{1}{3}\right) = \sqrt{\left(\frac{7}{3}-1\right)^2 + \left(\frac{7}{3}-1\right)^2 + \left(-\frac{1}{3}-1\right)^2}$$

$$= \frac{4}{\sqrt{3}}$$

E.g. A rectangular box without a lid is to be made from $12m^2$ of cardboard. Find the max vol of the box.

Soln:



$$Vol = xyz$$
$$SA = 2xz + 2yz + xy \leq 12m^2$$

Interior ($2xz + 2yz + xy < 12$):

$$\left.\begin{aligned} \frac{\partial V}{\partial x} &= yz = 0 \\ \frac{\partial V}{\partial y} &= xz = 0 \\ \frac{\partial V}{\partial z} &= xy = 0 \end{aligned}\right\}$$

If one of x, y, or z is 0, then the volume is 0.
$x, y, z \neq 0$
$\therefore$ No crit point within interior.

$$L = xyz - \lambda(2xz + 2yz + xy - 12)$$

$$\frac{\partial L}{\partial x} = yz - 2\lambda z - \lambda y = 0 \longrightarrow yz = \lambda(2z + y) \quad (1)$$

$$\frac{\partial L}{\partial y} = xz - 2\lambda z - \lambda x = 0 \longrightarrow xz = \lambda(2z + x) \quad (2)$$

$$\frac{\partial L}{\partial z} = xy - 2\lambda x - 2\lambda y = 0 \longrightarrow xy = \lambda(2x + 2y) \quad (3)$$

$$\frac{\partial L}{\partial \lambda} = -2xz - 2yz - xy + 12 = 0 \quad (4)$$

If we multiply ① by x, we get
$xyz = \lambda(2xz + xy)$.

If we multiply ② by y, we get
$xyz = \lambda(2yz + xy)$.

If we multiply ③ by z, we get
$xyz = \lambda(2xz + 2yz)$.

If $\lambda \neq 0$, we get:
1. $x = y$
2. $x = 2z$, $y = 2z$

Plugging $y = 2z$ and $x = 2z$ into ④,
we get:
$-4z^2 - 4z^2 - 4z^2 + 12 = 0 \rightarrow z^2 = 1 \rightarrow z = 1$
When $z = 1$, $x = y = 2$

$\therefore$ $(2,2,1)$ is a crit point and the max volume is $4m^3$.

3. Lagrange Multiplier With Multiple Constraints:

- If there are multiple constraints, $g_1(x) = c_1$, $g_2(x) = c_2, \ldots, g_n(x) = c_n$, we may construct the Lagrange function:

$$L(x, \lambda_1, \lambda_2, \ldots \lambda_n) = f(x) - \lambda_1(g_1(x) - c_1) - \ldots - \lambda_n(g_n(x) - c_n)$$

and then find all the crit points of L about λ and the constrained crit points of f.

- E.g. Find the extrema values of $f(x,y,z) = 3x - y - 3z$ subject to the constraints $x + y - z = 0$ and $x^2 + 2z^2 = 1$.

Soln:

1. $y = z - x$ (Bounded)
2. $x^2 + \frac{z^2}{1/2} = 1$ (Bounded)

$$L(x, y, z, \lambda, \mu) = 3x - y - 3z - \lambda(x + y - z) - \mu(x^2 + 2z^2 - 1)$$

① $\frac{\partial L}{\partial x} = 3 - \lambda - 2\mu x = 0$

② $\frac{\partial L}{\partial y} = -1 - \lambda = 0 \rightarrow \lambda = -1$

③ $\frac{\partial L}{\partial z} = -3 + \lambda - 4\mu z = 0$

④ $\frac{\partial L}{\partial \lambda} = -x - y + z = 0$

⑤ $\frac{\partial L}{\partial \mu} = -x^2 - 2z^2 + 1 = 0$

From 2, we calculated $\lambda=-1$.
Plugging that into 1 and 3, we get:

1 $3-(-1)-2x\mu=0$ 3 $-3+(-1)-4\mu z=0$

$4-2x\mu=0$ $-4-4z\mu=0$

$-2x\mu=-4$ $-4z\mu=4$

$x\mu=2$ $z\mu=-1$

$x=\frac{2}{\mu}$ $z=-\frac{1}{\mu}$

Plugging the x and z value into 5
we can find the value of μ.

5 $-x^2-2z^2+1=0$

$-x^2-2z^2=-1$

$x^2+2z^2=1$

$(\frac{2}{\mu})^2+2(-\frac{1}{\mu})^2=1$

$\frac{4}{\mu^2}+\frac{2}{\mu^2}=1$

$\frac{6}{\mu^2}=1$

$\mu^2=6$

$\mu=\pm\sqrt{6}$

When $\mu=+\sqrt{6}$, $x=\frac{2}{\sqrt{6}}$, $z=-\frac{1}{\sqrt{6}}$, $\overbrace{y=z-x}^{\text{From 4}}$

$=-\frac{3}{\sqrt{6}}$.

$(\frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$ is a crit point.

When $\mu = -\sqrt{6}$, $x = -\frac{2}{\sqrt{6}}$, $z = \frac{1}{\sqrt{6}}$, $y = z - x$
$= \frac{3}{\sqrt{6}}$

$\left(-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ is another crit point.

$f\left(\frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) = 2\sqrt{6}$ Max
$f\left(-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = -2\sqrt{6}$ Min

Note: I will cover multiple integrals in next week's notes.